The vacuum energy density in the teleparallel equivalent of general relativity

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Abstract

Considering the vacuum as characterized by the presence of only the gravitational field, we show that the vacuum energy density of the de Sitter space, in the realm of the teleparallel equivalent of general relativity, can acquire arbitrarily high values. This feature is expected to hold in the consideration of realistic cosmological models, and may possibly provide a simple explanation the cosmological constant problem.

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1 Introduction

Recent cosmological data indicate that the Universe is dominated by a form of exotic energy, the so called dark energy, which is expected to account for roughly seventy percent of the energy density of the universe. A strong, widely acceptable possibility is that this form of energy is due to Einstein's cosmological constant Λ (see, for example, Ref. [1]), which is presently considered to be of the order $\Lambda \simeq 10^{-56} cm^{-2}$. This fact has revived interest in the cosmological constant problem (CCP)[2, 3, 4, 5], which is considered by many as one of the most pressing conceptual problems in physics.

The CCP is characterized by the enormous difference in the expected values of the vacuum energy density that arises (i) from estimates in the realm of quantum field theory, namely, by considering the zero-point energy of quantum fields, and (ii) from estimates of the vacuum energy density in the context of Einstein's general relativity with a cosmological constant. If the space-time is endowed with a cosmological constant, the energy-momentum tensor acquires the additional term

$$T^{\Lambda}_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \ . \tag{1}$$

It is then asserted that the cosmological constant is expected to contribute to the vacuum energy density according to[2, 4, 5]

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \simeq 10^{-29} \ \frac{g}{cm^3} \ . \tag{2}$$

Alternatively, it is normally considered that the vacuum energy density acts like a cosmological constant.

Although it is very difficult to calculate the precise value of the vacuum energy density in quantum field theories, it is possible to make reasonable estimates of the contributions of the various fields. One concludes that the quantum vacuum energy density ρ_q is roughly of the order[4]

$$\rho_q \simeq 10^{97} \ \frac{g}{cm^3} \ , \tag{3}$$

which is 120 orders of magnitude larger than the estimate given by Eq. (1). Such huge difference constitutes the CCP. An usual formulation of the CCP is "Why is the vacuum energy, as given by Eq. (1), so small?"

In this article we suggest that the CCP may possibly have a simple explanation in the context of the Hamiltonian formulation of the teleparallel equivalent of general relativity (TEGR)[6, 7]. We will consider the definition of gravitational energy that arises in the realm of the TEGR, and that has been applied to several configurations of the gravitational field, yielding consistent results in all applications[7, 8]. In order to investigate the space-time vacuum energy density in the framework of the TEGR, we assume that the vacuum is characterized by the presence of only the gravitational field. Therefore the energy density of the vacuum is the energy density of the corresponding gravitational field. We will address the de Sitter space-time, which is a rather simplified model of a universe. Therefore the analysis to be described below should be understood as a mechanism that may be applied to the analysis of realistic cosmological models.

2 The gravitational energy definition of the TEGR

The TEGR is constructed out of tetrad fields $e_{a\mu}$, where a and μ are SO(3,1) and space-time indices, respectively, and provides a suitable framework where the notion of gravitational energy can be discussed and easily applied to any space-time geometry that admits a 3+1 foliation. The constraint equations of the Hamiltonian formulation of the TEGR are interpreted as energy, momentum and angular momentum equations for the gravitational field[7].

The Lagrangian density of the TEGR is given by

$$L(e) \;=\; -k\,e\left(\frac{1}{4}T^{abc}T_{abc}+\frac{1}{2}T^{abc}T_{bac}-T^aT_a\right)\,, \tag{4}$$

where $e=det(e^a\,_\mu)$, $T_{abc}=e_b\,^\mu e_c\,^\nu T_{a\mu\nu}$, $T_{a\mu\nu}=\partial_\mu e_{a\nu}-\partial_\nu e_{a\mu}$, and the trace of the torsion tensor is given by $T_b=T^a\,_{ab}$. The Hamiltonian is obtained by just rewriting the Lagrangian density in the form $L=p\dot{q}-H$. Since there is no time derivative of e_{a0} in (2.1), the corresponding momentum canonically conjugated Π^{a0} vanishes identically. Dispensing with surface terms the total Hamiltonian density reads[6]

$$H(e_{ai},\Pi^{ai}) = e_{a0}C^a + \alpha_{ik}\Gamma^{ik} + \beta_k\Gamma^k \ , \tag{5}$$

where $\{C^a, \Gamma^{ik}$ and $\Gamma^k\}$ constitute a set of primary constraints, and α_{ik} and β_k are Lagrange multipliers. Explicit details are given in Ref. [6]. The first term of the constraint C^a is given by a total divergence in the form $C^a = -\partial_k\Pi^{ak} + \cdots$. The equation $C^a = 0$ is interpreted as an energy-momentum equation for the gravitational field of the type $H^a - P^a = 0$, and we identify the total divergence on the three-dimensional spacelike hypersurface as the energy-momentum density of the gravitational field. The total energy-momentum is defined by

$$P^a = -\int_V d^3x\,\partial_i\Pi^{ai} \ , \tag{6}$$

where V is an arbitrary space volume. It is invariant under coordinate transformations on the spacelike manifold, and transforms as a vector under the global SO(3,1) group.

The gravitational energy E_g enclosed by an arbitrary space volume V is defined by[7]

$$E_g = -\int_V d^3x\,\partial_i\Pi^{(0)i} \ , \tag{7}$$

where $\Pi^{(0)i}$ is the momentum canonically conjugated to $e_{(0)i}$ (Latin indices $i, j, ...$ run from 1 to 3). It reads

$$\begin{aligned}\Pi^{ak} \;=\; & k\,e\Big\{g^{00}(-g^{kj}T^a{}_{0j} - e^{aj}T^k{}_{0j} + 2e^{ak}T^j{}_{0j}) \\ & +g^{0k}(g^{0j}T^a{}_{0j}+e^{aj}T^0{}_{0j}) + e^{a0}(g^{0j}T^k{}_{0j}+g^{kj}T^0{}_{0j}) - 2(e^{a0}g^{0k}T^j{}_{0j}+e^{ak}g^{0j}T^0{}_{0j}) \\ & -g^{0i}g^{kj}T^a{}_{ij} + e^{ai}(g^{0j}T^k{}_{ij} - g^{kj}T^0{}_{ij}) - 2(g^{0i}e^{ak} - g^{ik}e^{a0})T^j{}_{ji}\Big\} \ . \end{aligned} \tag{8}$$

With appropriate boundary conditions expression (7) yields the ADM energy. This expression satisfies the main requirements for a gravitational energy definition[7].

The torsion tensor cannot be made to vanish at a point by a coordinate transformation. This fact refutes the usual argument against the nonlocalizability of the gravitational energy, and which rests on the reduction of the metric tensor to the Minkowski metric tensor at a point in space-time by means of a coordinate transformation. In any small neighborhood of space the gravitational field can be considered constant and uniform. The principle of equivalence asserts that in such neighborhood it is always possible to choose a reference frame in which the gravitational effects are not observed. Thus in such reference frame we should not detect any form of gravitational energy. Therefore it is reasonable to expect that the localizability of the gravitational energy depends on the reference frame, but not on the coordinate system. In fact any other form of relativistic energy depends on the reference frame. It turns out that the gravitational energy definition given by Eq. (7) displays the feature discussed above, namely, it depends on the reference frame. More precisely, it depends on the choice of a global set of tetrad fields since the energy expression is not invariant under local SO(3,1) transformations of the tetrad field, but is invariant under coordinate transformations of the three-dimensional spacelike hypersurface (reference frames are better conceived in terms of fields of vector bases [9]).

3 The de Sitter space-time

The present investigation of the vacuum energy density is based on a previous analysis of the gravitational energy of the de Sitter space-time, and rests on two premises. **I.** If the quantum vacuum has indeed a physical reality, then such an enormous energy density must produce a gravitational field and a corresponding enormous gravitational energy density. If, in addition, the space-time is endowed with a positive cosmological constant, then such gravitational field is expected to give rise to the repulsive force typical of the de Sitter space-time. **II.** Since the vacuum is characterized by the existence of only the gravitational field, the energy density of the vacuum is the energy density of the (de Sitter) gravitational field. From this point of view, the energy density given by Eq. (2) does not represent the actual vacuum energy density induced by the cosmological constant.

We will show that in the framework of the TEGR it is possible to arrive at arbitrarily high values of the gravitational energy density if the space-time

is endowed with a positive cosmological constant. We will consider the de Sitter space-time as just a model for our discussion. In realistic cosmological models the feature to be described below should also take place. The metric tensor for de Sitter space-time is given by

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)dt^2 + \left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,, \tag{9}$$

where $R=\sqrt{\frac{3}{\Lambda}}$ (the discussion presented below can also be carried out in the context of the Schwarzschild-de Sitter solution). The de Sitter space-time will be described by the following set of tetrad fields,

$$e^a{}_\mu = \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 0 & \alpha\sin\theta\,\cos\phi & r\cos\theta\,\cos\phi & -r\sin\theta\,\sin\phi \\ 0 & \alpha\sin\theta\,\sin\phi & r\cos\theta\,\sin\phi & r\sin\theta\,\cos\phi \\ 0 & \alpha\cos\theta & -r\sin\theta & 0 \end{pmatrix} \,, \tag{10}$$

where

$$\alpha = \left(1-\frac{r^2}{R^2}\right)^{-\frac{1}{2}} \,.$$

The set of tetrad fields given above satisfy the conditions

$$e_{(i)j} = e_{(j)i} \,, \tag{11}$$

$$e_{(i)}{}^0 = 0 \,. \tag{12}$$

In a space-time determined by a global set of tetrad fields $e^a{}_\mu$ we may consider the existence of an underlying reference space-time with coordinates q^a that are anholonomically related to the physical space-time coordinates by means of the relation $dq^a = e^a{}_\mu dx^\mu$. Conditions (11) and (12) above on the tetrad fields establish a unique reference space-time that is neither related by a boost transformation, nor rotating with respect to the physical space-time[7]. Therefore we will evaluate the gravitational energy of the de Sitter space-time with respect to the frame defined by Eqs. (11) and (12).

If the tetrad fields satisfy the time gauge condition (12), then expression (7) reduces to (see Eq. (3.11) of Ref. [7])

$$-\int_V d^3x\,\partial_i\Pi^{(0)i} = \frac{1}{8\pi G}\int_V d^3x\partial_j(eT^j) = \frac{1}{8\pi G}\int_S dS_j(eT^j)\,. \qquad (13)$$

We have the definitions $e = \det(e_{(i)j})$, $T^i = g^{ik}T_k = g^{ik}e^{(m)j}T_{(m)jk}$, and $T_{(m)jk} = \partial_j e_{(m)k} - \partial_k e_{(m)j}$. All these field quantities are restricted to the three-dimensional spacelike hypersurface. As a consequence, the calculation of Eq. (7) out of tetrads given by Eq. (6) yields precisely the expression obtained in Ref. [8]. By integrating Eq. (7) over a surface S defined by $r = R$ we obtain the total energy[8]

$$E_{dS} = \frac{1}{8\pi G}\int_S d\theta d\phi\,(eT^1) = \frac{1}{G}R = \frac{1}{G}\sqrt{\frac{3}{\Lambda}}\,, \qquad (14)$$

and the average energy density

$$\frac{E_{dS}}{2\pi^2 R^3} = \frac{\Lambda}{6\pi^2 G}\,. \qquad (15)$$

In Eq. (14) we have

$$eT^1 = 2r\,\sin\theta\left[1 - \sqrt{1 - \frac{r^2}{R^2}}\right]\,. \qquad (16)$$

The gravitational energy density is given by $(1/8\pi G)\partial_r(eT^1)$. By integrating this expression in θ and ϕ we obtain the gravitational energy per unit radial distance $\varepsilon(r)$. Therefore upon integration in θ and ϕ and differentiation in r of Eq. (16) we arrive at[8]

$$E_{dS} = \int_0^R dr\varepsilon(r)\,, \qquad (17)$$

where

$$\varepsilon(r) = \frac{1}{G}\left[1 + \frac{2\beta^2 - 1}{\sqrt{1-\beta^2}}\right]\,, \qquad (18)$$

and $\beta^2 = r^2/R^2$.

A crucial point of the present analysis is that $\varepsilon(r)$ is clearly divergent in the limit $r \to R$, i.e., $\varepsilon(r) \to \infty$. This fact is an indication that the CCP may have an explanation in the present framework. In order to verify this issue, let us first note that $\varepsilon(r)\,dr$ is the gravitational energy contained within

the spherical shell of radius r and width dr. Since the area of such shell is $4\pi r^2$, the gravitational energy per unit volume is given by $\rho(r) = \varepsilon(r)/(4\pi r^2)$ (we arrive at precisely the same expression by a suitable coordinate transformation). We obtain the correct dimensions upon the introduction of the velocity of light c by means of the replacement $1/G \to c^4/G$. Therefore the gravitational energy density $\rho(r)$ is given by

$$\rho(r) = \frac{c^4}{G}\frac{1}{4\pi r^2}\left[1 + \frac{2\beta^2 - 1}{\sqrt{1-\beta^2}}\right] . \tag{19}$$

In order to compare Eq. (19) with expression (3) we use

$$\frac{c^4}{G} = 1.21 \times 10^{49}\, g\frac{cm}{s^2} = 0.135 \times 10^{29}\frac{g}{cm} .$$

Therefore we must find a radial position r of an observer such that

$$(0.135 \times 10^{29}\frac{g}{cm})\frac{1}{4\pi r^2}\left[1 + \frac{2\beta^2 - 1}{\sqrt{1-\beta^2}}\right] \simeq 10^{97}\ \frac{g}{cm^3} . \tag{20}$$

Replacing r^2 in the denominator of $1/(4\pi r^2)$ in Eq. (14) by $\beta^2 R^2 = \beta^2(1.73\times 10^{28} cm)^2$ we find

$$\frac{1}{\beta^2}\left[1 + \frac{2\beta^2 - 1}{\sqrt{1-\beta^2}}\right] \simeq 10^{127} \equiv n . \tag{21}$$

The equation above leads to a simple equation for β^2,

$$n^2\beta^4 + (4 - 2n - n^2)\beta^2 + 2n - 3 = 0 , \tag{22}$$

whose solutions are given by

$$\beta^2 = \frac{1}{2n^2}\left[-4 + 2n + n^2 \pm \left(n^4 - 4n^3 + 8n^2 - 16n + 16\right)^{\frac{1}{2}}\right] . \tag{23}$$

The approximate solutions of the equation above are $\beta^2 \simeq 1 - 1/n^2$, that yields

$$r \simeq R(1 - 10^{-254}) , \tag{24}$$

and $\beta^2 \simeq 2/n$, that implies

$$r \simeq 10^{-63} R \simeq 10^{-35} cm \; . \tag{25}$$

Therefore in order for an hypothetical observer in the de Sitter universe to be under the effect of the energy density given by Eq. (3), either its position r practically coincides with the cosmological horizon (r given by Eq. (24)) or the observer must stand extremely close to the origin of the coordinate system (r given by Eq. (25)).

4 Discussion

The gravitational energy density given by Eq. (19) may acquire arbitrarily high values, depending on the value of the coordinate r. We note that in quantum field theories the establishment of the ultraviolet cutoff that leads to the value given by Eq. (3) is an assumption related to the validity of quantum field theory up to the Planck scale, and is precisely required to avoid a divergence. In the present context of a classical, simplified cosmological model we do not yet dispose of a similar argument to assert that Eq. (19) acquires very large but finite values. The imposition of cutoffs is a device typical of a quantum field theories.

The de Sitter space-time describes an empty universe. Inspite of this fact, if we imagine an observer located at $r \simeq 0$ or at $r \simeq R$, such observer will experience an enormous gravitational energy density due to the cosmological constant. In a *realistic* cosmological model, with an arbitrary matter distribution, we expect a similar situation to hold, namely, we expect that there are regions in space where the vacuum energy density acquires values of the order of Eq. (3). Such energy density is expected to be related to the quantum vacuum energy density. According to this picture, the vacuum energy density is not homogeneous in space. However, the repulsive acceleration in the de Sitter space-time induced by the cosmological constant, $a = (1/3)\Lambda r$, also constitutes a *nonhomogeneous* characteristic of the space-time which, we believe, is intrinsically related to the above mentioned nonhomogeneity. Even though Eq. (19) holds for a quite simplified model of universe, the nonhomogeneity of the vacuum energy density (on very large scales of distance) *cannot* be ruled out on empirical grounds. On the contrary, this hypothesis may lead to new ideas about the relation between gravity and quantum

theory.

We note finally that the present analysis of gravitational energy in the de Sitter space is different from the one carried out by Abbot and Deser[10], who provided an expression for the gravitational energy *about* the de Sitter background, i.e., they calculated the energy of a field configuration that deviates from the de Sitter metric and vanish at infinity (this point is discussed in Ref. [8]).

In summary, we have presented a simple explanation to describe the emergence of extremely high values of the vacuum energy density in the universe. Such mechanism may explain the cosmological constant problem. The present result is achieved by reinterpreting the vacuum energy density as the gravitational energy density of the de Sitter space-time.

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References

[1] P. J. E. Peebles and B. Ratra, *The Cosmological Constant and Dark Energy*, preprint astro-ph/0207347.

[2] Ya. B. Zel'dovich, Usp. Fiz. Nauk. **95**, 209 (1986) [English translation in Sov. Phys. Usp. **11**, 381 (1968)].

[3] N. Straumann, *On the Cosmological Constant Problems and the Astronomical Evidence for a Homogeneous Energy Density with Negative Pressure*, preprint astro-ph/0203330 (Lecture given at the *Séminaire Poincaré*, Paris, March 2002).

[4] U. Ellwanger, *The Cosmological Constant*, preprint hep-th/0203252 (Lecture given at the XIV Workshop "Beyond the Standard Model", Bad Honef, March 2002).

[5] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[6] J. W. Maluf and J. F. da Rocha-Neto, Phys. Rev. D **64**, 084014 (2001).

[7] J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toríbio and K. H. Castello-Branco, Phys. Rev. D **65**, 124001 (2002).

[8] J. W. Maluf, J. Math. Phys. **37**, 6293 (1996).

[9] R. Aldrovandi, P. B. Barros and J. G. Pereira, Found. Phys. (2003), *The Equivalence Principle Revisited* [gr-qc/0212034].

[10] L. F. Abbot and S. Deser, Nucl. Phys. B **195**, 76 (1982).